HW1

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Problem 1

1. 0
2. No
3. 0

Notes on SPD Matrices, Inner Products, Norms, andMetrics

Problem 1: prove Fact 3 – If is SPD, then A is invertible, and is SPD too.

Proof: Let be a SPD matrix. Assume is not invertible, thus there exist vector such that .  
 in contradiction to the fact that is SPD. ()  
Therefore is invertible, meaning exists.

Let be an eigenvalue of , hence:  
According to fact 1, all eigenvalues of are positive. Therefore, all (the eigenvalues of ) are positive, and again from fact 1 we obtain that is SPD.

Problem 2: prove Fact 4 – Let be an SPD matrix. Then is an inner product.

Solution: Let Q be an SPD matrix and

We will show that the properties of definition 6 hold.

1. Let , then Let , then from SPD definition,

Problem 3: prove Fact 8 – Every norm induces a metric: .

Proof: Let , we will show that the properties of definition 11 hold.

1. Let then, from definition,

Let , then and from definition

Problem 4: prove Fact 10 – Let be an matrix and denote its Cholesky decomposition.  
 Then .

Proof: Let be an matrix and denote its Cholesky decomposition.

**Computer Exercise 1:**

The relevant code will be added separately in the submission page.

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c: A picture containing icon

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e:A picture containing text

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Notes on Convexity

Problem 1:

1. is convex:

Let assume that , so:

This is true because by definition and because we will get that .

1. is convex:

Let assume that , so:

1. is convex:

Let assume that , so:

1. is not convex:

For :

and because , is not convex.

1. is not convex:

For :

and because , is not convex.

Notes on Argmin and Argmax

Problem 1: Let be a function from some set into . Then

Proof: Let be a function from some set into .

Problem 2: For some monotonically non-increasing function, and some it holds that:

Proof: set as and as .

It is easy to see that .

But now: , while for value , and .

Therefore .

Problem 3: Let be a monotonically non-decreasing function. Let be a function we seek to maximize. Then, .

Proof: By the definition of argmax:

Problem 4: Let depend on . Show that .

Proof: By using Fact 5 for , and , we directly deduce that:

that is because that , and is monotonically increasing on [0, ∞].

Problem 5: Let and let depend on . Show that:

Proof: From Problem 4 we get that:

From Fact 1 we get that

Now, again from Fact 5, for monotonically increasing function we get that:

And again for monotonically increasing function we get that

And one last time, for monotonically increasing function we get that :

So overall we achieved that:

Notes on Linear Least Squares

Problem 1: Find

where

Proof: We will bring our problem to a least squares manner:

Thus, from linear least squares, the minimizer satisfies the following equation:

is always SPD, and the addition of positive values on the diagonal won^' t change that so the new matrix is invertible.

Problem 2: Find

Proof: as mentioned at the notes, , when   
and and therefore :

Meaning the two following problems are equal:

We know how to solve it by LS and the normal equations:

Because is orthogonal we get:

**Notes on Random Vectors**

Problem 1: satisfies finite additivity; namely, if is a finite collection of pairwise disjoint events then .

Proof:

Let us denote for every , and now:

Problem 2: Prove that

Proof:

Problem 3: Let be a RV whose codomain is , Find and where

1)

2)

Proof:

And because , for every , , and therefore:

And

And because , there is no that , and therefore:

And

Problem 4: Is the following function,, a CDF of some RV?

Proof:

No, we will show that is not right-continuous.

A function is right-continuous at C if

Therefore,

Means does not holds right-continuous property of CDF is not CDF.

Problem 5: Let be a continuous -dimensional RV.

1. Let . Find
2. Give an example for such that contains at least one element of and .
3. Let be a countable collection of nonempty pairwise disjoint subsets of ( i.e., for every and whenever ). Give an example for such a collection where, in addition,

Is zero.

Proof:

Let us denote , now:

Because X is continuous RV then the CDF is continuous (definition 9). Thus,

Now, .

1. For , denote , therefore:
2. For , denote , therefore:

We can notice that for every : and . Now, for: there exist such that , and therefore , so:

Problem 6: Let stand for the result of rolling a fair die. Thus, for and   
and Draw F(X) and .

Proof:

Chart, line chart

Description automatically generatedChart

Description automatically generated

Problem 7: Write a similar expression for .

Proof:

Problem 8: Let stand for the result of rolling a fair die (i.e., is distributed uniformly on Show that .

Proof:

Problem 9: let . Then the matrix is symmetric; namely, .

Proof:

Problem 10: Show that .

Proof:

Problem 11: Show that (i.e., the element in the row and column of )  
 is

Proof:­

Denote RV (discrete or continuous – we will see it results in the same ☺)

such that

Problem 12: Show that is symmetric, i.e., show that = .

Proof: We will show that

For discrete RVs:

For continuous RVs:

Now, from as we saw in problem 11:

Problem 13: Show that (i.e., the element in therow and column of ) is

Proof:­

Similar to problem 11, denote RV.

such that

Problem 14: Show that is the variance of .

Proof: From definition,

From problem 13, we obtain:

Problem 15: Show that is symmetric, i.e., show that = .

Proof: We will show that

Denote , hence we need to prove:

But we already proved it in problem 12 for discrete and continuous RVs

Now from problem 13:

Problem 16: Prove:

Proof:

Problem 17: Prove:

Proof:

Problem 18: Let affine transformation. Prove:

Proof:

Problem 19: Let be an n-dimensional RV with mean and covariance .

Find:

Proof:

Denote:

the affine transformation of .

From fact 10:

Problem 20:

Proof:

We’ll prove:

Problem 21: Show that

Proof:

Problem 22: prove Fact 13:

1. If and are orthogonal RVs, then the correlation matrix of is
2. If and are uncorrelated RVs, then the covariance matrix of is

Proof:

1. and are orthogonal RVs, hence from definition 37:

Recall from definition 28 that

Finally, we saw in fact 9 that the correlation matrix of Z is

1. and are uncorrelated RVs, hence from definition 38: are orthogonal, and again from definition 37 we obtain:

Recall from definition 28 and problem 17 that

Finally, we saw in fact 9 that the covariance matrix of Z is

Problem 23: prove Fact 14:

Proof:

X and Y are independent RVs, hence by definition .  
We need to show that X and Y are uncorrelated, meaning:

For discrete RVs:

For continuous RVs:

Problem 24: Let be a two-dimensional random vector taking values in .   
The probability mass function (pmf) of X is given by:

1. Find the marginals, .
2. Find the 2D mean vector, .
3. Find the 2-by-2 correlation matrix, .
4. Find the 2-by-2 covariance matrix, .
5. Are and independent?
6. Are and correlated?

Proof:

1. and are not independent since

Meaning, and are not uncorrelated and are correlated.

Problem 25: Let be a two-dimensional random vector taking values in .   
The probability mass function (pmf ) of X is given by:

1. Find the marginals, .
2. Find the 2D mean vector, .
3. Find the 2-by-2 covariance matrix, .
4. Are and independent?
5. Are and uncorrelated?

Let and let

1. Find the pmf of .
2. Find the marginal pmf of and the marginal pmf of .
3. Find the 2D mean vector of , .
4. Find the 2-by-2 covariance matrix of Y, .
5. Are and independent?
6. Are and uncorrelated

Proof:

Similarly, we will obtain:

1. and are independent since

Meaning, and are uncorrelated.

1. We can notice that and
2. and are not independent since

Meaning, and are not uncorrelated and are correlated.

Problem 26:

Proof: We'll prove:

Let us notice that:

Problem 27: Let be some bounded hyper-rectangle in .

1. Let

Show that

1. Let

Show that

Proof:

1. For

and

we get that:

and now from Fact 18 we get that: .

1. For bounded hyper-rectangle we can see that for :

And:

Therefore: .

Problem 28: Show that is -field

Proof:

We will show that:

1. is closed under complementation
2. is closed under countable unions
3. From definition of power-set:
4. Mark For ,

Now, from power-set definition we get that:.

1. Mark For

Now, from power-set definition we get that:.

Problem 29: Let be a -field.

1. Show that .
2. Using De Morgan's laws, show that is closed under countable intersections. Namely, if are in then

Proof:

1. Because , we get from property (2) of -field that:
2. De Morgan's laws state that Therefore:

For all , from property (2)

from property (3) , and therefore: .