HW1

Ilana Pervoi, Pan Eyal

Problem 1

1. 0
2. No
3. 0

Notes on SPD Matrices, Inner Products, Norms, andMetrics

Problem 1: fact 3: If is SPD, then A is invertible, and is SPD too.

Proof: Let be a SPD matrix. Assume is not invertible, thus there exist vector such that .  
 in contradiction to the fact that is SPD. ()  
Therefore is invertible, meaning exists.

Let be an eigenvalue of , hence:  
According to fact 1, all eigenvalues of are positive. Therefore, all (the eigenvalues of ) are positive, and again from fact 1 we obtain that is SPD.

Problem 2: fact 4: Let be an SPD matrix. Then is an inner product.

Proof: Let Q be an SPD matrix and

We will show that the properties of definition 6 hold.

1. Let , then Let , then from SPD definition,

Problem 3: fact 8: Every norm induces a metric: .

Proof: Let , we will show that the properties of definition 11 hold.

1. Let then, from definition,

Let , then and from definition

Problem 4: fact 10: Let be an matrix and denote its Cholesky decomposition.  
 Then .

Proof: Let be an matrix and denote its Cholesky decomposition.

**Computer Exercise 1:**

a:Shape

Description automatically generated b:A picture containing icon

Description automatically generated

c: A picture containing icon

Description automatically generatedd: A picture containing text

Description automatically generated

e:A picture containing text

Description automatically generated

Notes on Convexity

Problem 1:

1. is convex:

Let assume that , so:

This is true because by definition and because we will get that .

1. is convex:

Let assume that , so:

1. is convex:

Let assume that , so:

1. is not convex:

For :

and because , is not convex.

1. is not convex:

For :

and because , is not convex.

Notes on Argmin and Argmax

Problem 1: Let be a function from some set into . Then

Proof: Let be a function from some set into .

Problem 2: For some monotonically non-increasing function, and some it holds that:

Proof: set as and as .

It is easy to see that .

But now: , while for value , and .

Therefore .

Problem 3: Let be a monotonically non-decreasing function. Let be a function we seek to maximize. Then, .

Proof: By the definition of argmax:

Problem 4: Let depend on . Show that .

Proof: By using Fact 5 for , and , we directly deduce that:

that is because that , and is monotonically increasing on [0, ∞].

Problem 5: Let and let depend on . Show that:

Proof: From Problem 4 we get that:

From Fact 1 we get that

Now, again from Fact 5, for monotonically increasing function we get that:

And again for monotonically increasing function we get that

And one last time, for monotonically increasing function we get that :

So overall we achieved that:

Notes on Linear Least Squares

Problem 1: Find

where

Proof: We will bring our problem to a least squares manner:

Thus, from linear least squares, the minimizer satisfies the following equation:

is always SPD, and the addition of positive values on the diagonal won^' t change that so the new matrix is invertible.

Problem 2: Find

Proof: as mentioned at the notes, , when   
and and therefore :

Meaning the two following problems are equal:

We know how to solve it by LS and the normal equations:

Because is orthogonal we get:

**Notes on Random Vectors**

Problem 1: satisfies finite additivity; namely, if is a finite collection of pairwise disjoint events then .

Proof:

Let us denote for every , and now:

Problem 2: Prove that

Proof:

Problem 3: Let be a RV whose codomain is , Find and where

1)

2)

Proof:

And because , for every , , and therefore:

And

And because , there is no that , and therefore:

And

Problem 4: Is the following function,, a CDF of some RV?

Proof:

No, we will show that is not right-continuous.

A function is right-continuous at C if

Therefore,

Means does not holds right-continuous property of CDF is not CDF.

Problem 5: Let be a continuous -dimensional RV.

1. Let . Find
2. Give an example for such that contains at least one element of and .
3. Let be a countable collection of nonempty pairwise disjoint subsets of ( i.e., for every and whenever ). Give an example for such a collection where, in addition,

Is zero.

Proof:

Let us denote , now:

Because X is continuous RV then the CDF is continuous (definition 9). Thus,

Now, .

1. For , denote , therefore:
2. For , denote , therefore:

We can notice that for every : and . Now, for: there exist such that , and therefore , so:

Problem 6: Let stand for the result of rolling a fair die. Thus, for and   
and Draw F(X) and .

Proof:

Chart, line chart

Description automatically generatedChart

Description automatically generated

Problem 7: Write a similar expression for .

Proof:

Problem 8: Let stand for the result of rolling a fair die (i.e., is distributed uniformly on Show that .

Proof:

Problem 9: let . Then the matrix is symmetric; namely, .

Proof:

Problem 10: Show that .

Proof: